Discontinuous velocity profiles for the Orr–Sommerfeld equation

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Simple ideas of dimensional analysis and of limiting cases are used to elucidate the stability characteristics of a steady basic parallel flow of a viscous incompressible fluid. The principal result is that the stability characteristics of a smoothly varying velocity profile for wave disturbances of small wave-number can be found by use of a *discontinuous* velocity profile. The boundary conditions for a disturbance at a discontinuity of the basic flow are derived, and are used to find the stability characteristics of broken-line representations of the half-jet and jet. These findings are in agreement with previous ones.

1. Introduction

Most work on the Orr–Sommerfeld equation involves elaborate analysis (cf. Lin 1955). It seems desirable to devise simpler, though possibly less informative, methods in order to understand the nature of instability more directly, to teach the analysis to graduate classes, and to make it accessible to specialists in other fields. Such methods could aid new calculations of stability characteristics when the influence of variation of density or viscosity, of magnetic fields, or of rotating systems is considered. In this paper dimensional analysis and limiting cases of small Reynolds number or small wave-number will be used. The principal result is an interpretation and justification of the use of *discontinuous* velocity profiles for finding stability characteristics of a parallel flow of a viscous incompressible fluid.

Recently, the stability characteristics of an unbounded jet (Tatsumi & Kakutani 1958; Howard 1959) and half-jet (Tatsumi & Gotoh 1960) have been found by use of expansions in the wave-number, valid when that number is small. In these papers the growth rate of a small wave disturbance was expressed essentially as an explicit algebraic function of integrals of the velocity distribution function of the basic flow. Because only integrals of the velocity distribution are involved, these methods could be used for discontinuous velocity distributions. For example, a piecewise-constant distribution could be used to approximate the growth rate of a smoothly varying distribution as closely as desired. The apparent physical absurdity of a discontinuity of velocity in a viscous fluid will be shown in this paper to arise from a misinterpretation of a mathematical method of approximation valid for small wave-numbers. In fact, Esch (1957) has already used the basic velocity distribution

$$w_*(y_*) = \begin{cases} Vy_*/|y_*| & (|y_*| > L), \\ Vy_*/L & (|y_*| < L), \end{cases} \tag{1}$$

with two discontinuities of viscous stress to represent the half-jet. He assumed three, and then proved the fourth, of the boundary conditions at each of the discontinuities $y_* = \pm L$. These conditions were used to join up the solutions in the regions $y_* > L$, $L > y_* > -L$, $-L > y_*$ and get the eigenvalue relation for the growth rate. In § 2 we shall prove Esch's four conditions at a discontinuity of basic velocity as well as stress. In §§ 3, 4 these conditions will be applied to broken-line representations of the velocity profiles of the half-jet and jet. The results are in agreement with those of Tatsumi & Gotoh (1960) and of Tatsumi & Kakutani (1958, 1960).

First, we shall summarize the eigenvalue problem used to derive the stability characteristics, and introduce some ideas of dimensional analysis and of limiting cases. When a basic flow with velocity $w_*(y_*)$ in the x_*-direction is bounded by rigid walls at $y_* = y_{*1}, y_{*2}$ (where y_{*1} and/or y_{*2} may be infinite if the flow is unbounded), it is known (cf. Lin 1955) that the stability characteristics can be found from the eigenvalues of the Orr–Sommerfeld equation

$$(D_*^2-\alpha_*^2)^2\phi_* = (i\alpha_*/\nu)\{(w_*-c_*)(D_*^2-\alpha_*^2)\phi_*-(D_*^2w_*)\phi_*\} \tag{2}$$

and the boundary conditions

$$\alpha_*\phi_* = 0 = D_*\phi_* \quad \text{for} \quad y_* = y_{*1}, y_{*2}. \tag{3}$$

An asterisk is used as a subscript to denote dimensional parameters and variables; $D_* \equiv d/dy_*$; and ν is the kinematic viscosity of the fluid. The solution of the eigenvalue problem gives the stream function of a perturbation of the basic flow, assumed to be of the form

$$\psi'_* = \phi_*(y_*)\exp\{i\alpha_*(x_*-c_*t_*)\}. \tag{4}$$

Thus, the flow is stable or unstable to small disturbances of positive wave-number α_* accordingly as the imaginary part c_{*i} of the complex velocity $c_*(\alpha_*)$ is negative or positive, respectively.

If the velocity distribution $w_*(y_*)$ is dimensionally characterized by some length scale L and velocity scale V, the Reynolds number may be defined as

$$R = VL/\nu. \tag{5}$$

With dimensionless variables $y \equiv y_*/L$, $w(y) \equiv V^{-1}w_*(y_*)$, $\phi(y) \equiv V^{-1}L^{-1}\phi_*(y_*)$ and parameters $\alpha \equiv \alpha_*L$, $c \equiv V^{-1}c_*$, the Orr–Sommerfeld equation becomes

$$(D^2-\alpha^2)^2\phi = i\alpha R\{(w-c)(D^2-\alpha^2)\phi-(D^2w)\phi\}, \tag{6}$$

and the boundary conditions become

$$\alpha\phi = 0 = D\phi \quad \text{for} \quad y = y_1, y_2. \tag{7}$$

The solutions of the equation are integral functions of y, c, α^2 and αR (or R/α). This leads to an eigenvalue relation of the form $F(c, \alpha^2, R/\alpha) = 0$, where F is an integral function of c, α^2, R/α. This may be written as

$$c = c(\alpha, R/\alpha), \tag{8}$$

which function may be multi- or single-valued or undefined for given values of α, R/α.

We shall now take the argument applied by Drazin & Howard (1961) to stability of an inviscid fluid and generalize it for a viscous fluid. Let us consider unbounded flows only, with derivatives of the velocity distribution function tending to zero at infinity. These flows are of two basic types. For the half-jet type, $w_*(-\infty) \neq w_*(\infty)$. We may choose the origin of velocity (by making a Galilean transformation if necessary) so that $w_*(-\infty) = -w_*(\infty)$. This may change c_r, but not c_i, and hence not the growth rate of the disturbance. We then take $V = w_*(\infty)$. For the jet type, $w_*(-\infty) = w_*(\infty)$; and we choose the origin of velocity so that $w_*(\infty) = 0$. We shall not specify V for this case generally. These choices give

$$w(y) \rightarrow \begin{Bmatrix} y/|y| & \text{(half-jet type),} \\ 0 & \text{(jet type),} \end{Bmatrix} \tag{9}$$

as $y \rightarrow \pm\infty$.

Let us suppose $L \rightarrow 0$ for a fixed function $w(y)$. Then

$$\begin{aligned} w_*(y_*) &= Vw(y) = Vw(y_*/L) \\ &\rightarrow \begin{Bmatrix} Vy_*/|y_*| & \text{(half-jet type),} \\ 0 & \text{(jet type),} \end{Bmatrix} \end{aligned} \tag{10}$$

because $y_*/L \rightarrow \pm\infty$ accordingly as $\pm y_* > 0$, respectively. Also $\alpha \rightarrow 0$, $R/\alpha =$ const. as $L \rightarrow 0$ for fixed α_*. Therefore $c \rightarrow c(0, R/\alpha) \equiv f(V/\alpha_* \nu)$, say, and

$$c_* \rightarrow Vf(V/\alpha_* \nu). \tag{11}$$

(We have assumed that the model is physically reasonable so that these limits may exist.)

Consider the two limits: (*a*) $\alpha_* \rightarrow 0$ for fixed $R/\alpha = V/\alpha_* \nu$ and L; (*b*) $L \rightarrow 0$ for fixed α_* and V/ν. Each limit, (*a*) or (*b*), gives $\alpha \rightarrow 0$ for fixed R/α. Therefore each limit for fixed $w(y)$ gives the same limiting form $f(R/\alpha)$ of $c(\alpha, R/\alpha)$ as $\alpha \rightarrow 0$ for fixed R/α. Therefore the stability characteristics $c_* \rightarrow Vf(V/\alpha_* \nu)$ are valid for both the limiting profile w_* when $L \rightarrow 0$ and the original w_* when $\alpha_* \rightarrow 0$. The limiting profile w_* when $L \rightarrow 0$ may have discontinuities or discontinuities of its derivatives. It will be shown that such profiles may be used to find the stability characteristics of smoothly varying profiles as $\alpha_* \rightarrow 0$ in this way.

These arguments show that all profiles of the half-jet type have the same stability characteristics as $\alpha \rightarrow 0$. These characteristics will be found directly in § 3 by use of the profile $w_* = Vy_*/|y_*|$ to determine $f(V/\alpha_* \nu)$. Our results agree with those of Tatsumi & Gotoh (1960).

For profiles of the jet type, $w_* \rightarrow 0$ as $L \rightarrow 0$. If $w_* = 0$, there is no non-zero eigensolution of the Orr–Sommerfeld equation which vanishes at infinity. However,

$$\phi_* = \text{const.}, \quad c_* = -i\alpha_* \nu \tag{12}$$

is a solution bounded at infinity. If this is an acceptable limit of the eigensolution, then

$$c \rightarrow -i\alpha/R \quad \text{as} \quad \alpha \rightarrow 0 \quad \text{for fixed } R/\alpha. \tag{13}$$

(Tatsumi & Kakutani (1958, equations (5.3), (5.8), (6.9)) show that in fact this is the proper limit, with $\phi \sim \text{const.} \times \exp\{-[\alpha^2 + i\alpha R(w-c)]^{\frac{1}{2}} |y|\}$ as $\alpha \rightarrow 0$, the square-root having non-negative real part.) In §4 we use the broken-line representation

$$w_* = \begin{cases} 0 & (|y_*| > H), \\ V & (|y_*| < H), \end{cases} \tag{14}$$

of the jet to find a better approximation than (13). A lower branch of the curve of neutral stability $c_i(\alpha, R/\alpha) = 0$ is found; it is of the same form as found by Tatsumi & Kakutani (1958) and Howard (1959) for the smoothly varying jet with $w_* = V\,\mathrm{sech}^2\,(y_*/L)$. A second branch of similar form also is found.

The above arguments are as valid for flows with one finite boundary as for flows with none. However, for flows with two finite boundaries there is an imposed length scale, namely, the distance between the boundaries, which cannot tend to zero without removing the field of flow.

If we let $V \to 0$ for a fixed function $w(y)$, then $w_* \to 0$. Therefore

$$c_* \to \alpha_* \nu g(\alpha_* L)$$

by dimensional analysis. Therefore $c \sim (\alpha/R)\,g(\alpha)$ as $R \to 0$ for fixed α. If we may suppose that the shape of the profile w_* is unimportant as $V \to 0$, we may deduce from equation (12) that $g(\alpha) = -i$, and

$$c \sim -i\alpha/R \tag{15}$$

as $R \to 0$ for fixed α.

Having examined the limit $\alpha_* \to 0$, let us put $\alpha_* = 0$. Then the stream function of the disturbance $\psi'_* = \phi_*(y_*)\exp(-\sigma_* t_*)$, for $\sigma_* \equiv i\alpha_* c_*$ need not vanish as $\alpha_* \to 0$. Then the longitudinal and lateral velocity components of the disturbance are respectively

$$u'_* = \partial\psi'_*/\partial y_* = (D_*\phi_*)\exp(-\sigma_* t_*), \quad v'_* = -\partial\psi'_*/\partial x_* = 0;$$

it can be seen that the velocity of the disturbance is parallel to the basic flow.

In this case with $\alpha_* = 0$ the Orr–Sommerfeld equation is

$$-\sigma_* D_*^2 \phi_* = \nu D_*^4 \phi_*.$$

Therefore

$$\phi_* \propto \begin{cases} 1,\ y_*,\ \exp[\pm i y_*(\sigma_*/\nu)^{\frac{1}{2}}] & (\sigma_* \neq 0), \\ 1,\ y_*,\ y_*^2,\ y_*^3 & (\sigma_* = 0). \end{cases}$$

If $y_{*1} = -L$, $y_{*2} = L$, the eigensolution satisfying boundary conditions (7) is either

$$\phi_* \propto \sin[(\sigma_*/\nu)^{\frac{1}{2}} y_*], \quad \alpha_* = 0, \quad \sigma_*/\nu = (n\pi/L)^2 \quad (n = 0, \pm 1, \pm 2, \ldots), \tag{16}$$

or

$$\phi_* = \text{const.}, \quad \alpha_* = 0, \quad \sigma_* \text{ indeterminate.} \tag{17}$$

For the former mode (16), $c = -i(n\pi)^2 R/\alpha$. If y_{*1} and/or y_{*2} is infinite, the only eigensolution is (17).

In contrast to this, the Navier–Stokes equations for parallel flow reduce to the diffusion equation

$$\partial u_*/\partial t_* = \nu\,\partial^2 u_*/\partial y_*^2,$$

whose solution for bounded flow is

$$u_*(y_*, t_*) = \sum_{n=1}^{\infty} b_{*n} \sin(n\pi y_*/L)\exp(-n^2\pi^2 t_* \nu/L^2),$$

where

$$b_{*n} \equiv \frac{1}{2\pi}\int_{-L}^{L} u_*(y_*, 0)\sin(n\pi y_*/L)\,dy_*;$$

and for unbounded flow the solution is

$$u_*(y_*, t_*) = \int_{-\infty}^{\infty} b_*(n_*) \exp\,(in_* y_* - n_*^2 t_* \nu)\, dn_*,$$

where
$$b_*(n_*) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} u_*(y_*, 0) \exp\,(-in_* y_*)\, dy_*.$$

However, the above solutions of the Orr–Sommerfeld and Navier–Stokes equations differ when the flow is unbounded (or semi-bounded, similarly). The difference occurs because no enumerable set of eigenfunctions of the Orr–Sommerfeld equation can form a base for solutions of the Navier–Stokes equations. In fact, a Fourier integral of solutions of the Orr–Sommerfeld can satisfy the boundary conditions though no single solution can.

In our search for simplicity we shall not comment further on the relation of the initial-value problem to that of separate wave components. We have considered the special case $\alpha_* = 0$ to see if it can shed some light on the solution of the Orr–Sommerfeld equation. Now all solutions of the diffusion equation are stable, so it might be concluded that those of the Orr–Sommerfeld equation give $c_{*i} \leqslant 0$ as $\alpha_* \to 0$ for fixed ν. This conclusion contradicts a result of Tatsumi & Gotoh (1960). The discrepancy appears to be due to the breakdown of the assumption of the existence of a steady parallel flow $w_*(y_*)$, made in the derivation of the Orr–Sommerfeld equation. As Tatsumi & Gotoh pointed out, when R or α is small, no unbounded steady flow is even approximately parallel in a characteristic length α^{-1}. We may add that the characteristic rate of change ν/L^2 of unbounded parallel flow is not very much less than the change $\alpha_* c_*$ of a small disturbance if R or α is small. In that case, the Orr–Sommerfeld equation has no physical relevance. However, mathematical knowledge of the limiting forms of the eigensolutions is nonetheless useful as a means to find the eigensolutions at greater values of R, α_* for which the Orr–Sommerfeld equation does describe approximately a real perturbation.

2. The boundary conditions for the disturbance at a discontinuity of the basic flow

In deducing the Orr–Sommerfeld equation (cf. Lin 1955), it is found that the equations of motion give

$$u'_* = (D_*\phi_*) \exp\{i\alpha_*(x_* - c_* t_*)\}, \quad v'_* = -i\alpha_*\phi_* \exp\{i\alpha_*(x_* - c_* t_*)\},$$

and, for the pressure of the disturbance,

$$p'_* = \rho\{(D_* w_*)\,\phi_* - (w_* - c_*)\,D_*\phi_* - i\nu\alpha_*^{-1}(D_*^2 - \alpha_*^2)\,D_*\phi_*\} \times \exp\{i\alpha_*(x_* - c_* t_*)\},$$

where ρ is the density of the fluid. Therefore the components of the stress tensor of the basic flow and the disturbance superposed are

$$p_{*xx} = \rho\{(w_* - c_*)\,D_*\phi_* - (D_* w_*)\,\phi_* + i\nu\alpha_*^{-1}(D_*^2 + \alpha_*^2)\,D_*\phi_*\} \times \exp\{i\alpha_*(x_* - c_* t_*)\},$$

$$p_{*xy} = p_{*yx} = \rho\nu D_* w_* + \rho\nu\{(D_*^2 + \alpha_*^2)\,\phi_*\} \exp\{i\alpha_*(x_* - c_* t_*)\},$$

$$p_{*yy} = \rho\{(w_* - c_*)\,D_*\phi_* - (D_* w_*)\,\phi_* + i\nu\alpha_*^{-1}(D_*^2 - 3\alpha_*^2)\,D_*\phi_*\} \times \exp\{i\alpha_*(x_* - c_* t_*)\}.$$

For stability of fluid at rest under the influence of gravity, Harrison (1908) supposed that these velocity and stress components were continuous at an interface where the density and viscosity were discontinuous. This method has also been used by Lock (1954) to deal with a basic flow under gravity with discontinuous viscosity and density distributions but continuous velocity and stress. For a basic flow (not under gravity) with continuous velocity but discontinuous stress, Esch (1957) implicitly assumed that ϕ_*, $D_*\phi_*$, $D_*^2\phi_*$ were continuous, and then deduced the 'jump' of $D_*^3\phi_*$ at the discontinuity of basic stress as follows. If the Orr–Sommerfeld equation is integrated from $y_{*0}-\epsilon$ to $y_{*0}+\epsilon$ across the discontinuity of D_*w_* at $y_*=y_{*0}$, we find

$$\left[D_*^3\phi_*-2\alpha_*^2D_*\phi_*\right]_{y_{*0}-\epsilon}^{y_{*0}+\epsilon}+\alpha_*^4\left[D_*^{-1}\phi_*\right]_{y_{*0}-\epsilon}^{y_{*0}+\epsilon}$$
$$=i\alpha_*\nu^{-1}\left[(w_*-c_*)D_*\phi_*-(D_*w_*)\phi_*\right]_{y_{*0}-\epsilon}^{y_{*0}+\epsilon}-\left[D_*^{-1}(w_*-c_*)\phi_*\right]_{y_{*0}-\epsilon}^{y_{*0}+\epsilon}.$$

Therefore

$$[D_*^3\phi_*+i\alpha_*\nu^{-1}(D_*w_*)\phi_*]=0 \tag{18}$$

as $\epsilon\to 0$, where square brackets are used to denote the 'jump', or difference across the discontinuity, of their contents. We shall develop this argument by showing that ϕ_*, $D_*\phi_*$ are continuous if w_* is piecewise continuously differentiable, and that discontinuous profiles may be used to approximate the eigenvalue c_* for a smoothly varying velocity profile at small wave-numbers. In view of the unphysical nature of the use of the discontinuous profiles, it seems wisest to abandon consideration of continuity of velocity or stress when seeking the approximate solution of the Orr–Sommerfeld equation. (Another reason is that that equation itself is derived by the approximation that the basic flow is an exact solution of the Navier–Stokes equations.) The Orr–Sommerfeld equation has solutions in simple terms of known functions in two cases: the solutions are exponential if $w_*(y_*)$ is constant, and involve Airy and exponential functions if $w_*(y_*)$ is linear. Our aim is merely to use these simple solutions to compute approximate values of c_* for curved profiles, as Rayleigh did in 1880 (cf. Rayleigh 1945, ch. 21) for an inviscid fluid.

Successive integrals of the Orr–Sommerfeld equation are

$$D_*^3\phi_*-i\alpha_*\nu^{-1}\{(w_*-c_*)D_*\phi_*-(D_*w_*)\phi_*\}$$
$$=2\alpha_*^2D_*\phi_*-\alpha_*^4D_*^{-1}\phi_*-i\alpha_*^3\nu^{-1}D_*^{-1}(w_*-c_*)\phi_*, \tag{19}$$

$$D_*^2\phi_*+i\alpha_*\nu^{-1}(w_*-c_*)\phi_*=2\alpha_*^2\phi_*-\alpha_*^4D_*^{-2}\phi_*$$
$$+i\alpha_*\nu^{-1}\{2D_*^{-1}(w_*-c_*)D_*\phi_*-\alpha_*^2D_*^{-2}(w_*-c_*)\phi_*\}, \tag{20}$$

$$D_*\phi_*=2\alpha_*^2D_*^{-1}\phi_*-\alpha_*^4D_*^{-3}\phi_*+i\alpha_*\nu^{-1}\{2D_*^{-2}w_*D_*\phi_*$$
$$-2c_*D_*^{-1}\phi_*-\alpha_*^2D_*^{-2}(w_*-c_*)\phi_*\}. \tag{21}$$

These equations suggest that $D_*^4\phi_*$, $D_*^3\phi_*$, $D_*^2\phi_*$ become discontinuous when a smoothly varying function $w_*(y_*)$ tends to a discontinuous one for a fixed function $w(y)$, because these differentials equal sums involving $D_*^2w_*$, D_*w_*

and w_*. However, $D_*\phi_*$, ϕ_* should be continuous, because they depend on w_* through its integrals only.

To prove this, suppose parameters α_*^2, $\alpha_*\nu^{-1}$ and the function $w(y)$ are fixed and non-zero as $L \to 0$. Further, if the flow is unbounded, suppose that all derivatives of $w(y)$ vanish at infinity, so that w is bounded.

Integrability or boundedness (though not continuity or differentiability) of a function of y implies the same of the corresponding function of y_* in the limit $L \to 0$. Therefore $\phi_*(y_*)$, like $\phi(y)$, is bounded. It follows that all terms, except $D_*^{-2}w_*D_*\phi_*$ possibly, of the right-hand side of equation (21) are continuous functions of y_* in the limit $L \to 0$. To show that this term also is continuous, we need only note that

$$|D_*^{-1}w_*D_*\phi_*| = V\,|D^{-1}wD\phi| \leqslant VD^{-1}\,|w|\,|D\phi| \leqslant V\big[D^{-1}\,|w|\,|D\phi|\big]_{y_1}^{y_2},$$

which is bounded, because $|w|$ and $\big[D^{-1}\,|D\phi|\big]_{y_1}^{y_2}$ are. Thus, the right-hand side of equation (21), and therefore $D_*\phi_*$, is continuous. The integral of equation (21) shows that ϕ_* is continuous. Finally, the right-hand sides of equations (19) and (20) give the jumps of $D_*^3\phi_*$ and $D_*^2\phi_*$ as $L \to 0$. Thus,

$$[\phi_*] = 0,$$

$$[D_*\phi_*] = 0,$$

$$[D_*^2\phi_* + i\alpha_*\nu^{-1}(w_* - c_*)\,\phi_*] = 0,$$

$$[D_*^3\phi_* - i\alpha_*\nu^{-1}\{(w_* - c_*)\,D_*\phi_* - (D_*w_*)\,\phi_*\}] = 0.$$

These conditions are valid as $L \to 0$ for bounded α_*, $\alpha_*\nu^{-1}$, in the sense that the resultant limiting profile $w_*(y_*)$ will give the same stability characteristics as the smoothly varying profile $w(y)$ does for small α.

These conditions may be applied to any flow, bounded, semi-bounded or unbounded, with only one length scale. However, bounded flows have an imposed length scale, $y_{*2} - y_{*1}$, which cannot tend to zero without the field of flow vanishing. When the basic flow has two length scales, we may suppose H is characteristic of distances between regions of shear of order V/L; in particular, we may take $H \equiv y_{*2} - y_{*1}$ for bounded flows. Then the above boundary conditions are still valid as $L \to 0$ in the above sense. Note that ν only occurs in the combination $\alpha_*\nu^{-1}$, so that it is possible to let $\alpha_*H \to 0$ for fixed $\alpha_*H^2V\nu^{-1}$ in this case.

It is known (cf. Lin 1955) that bounded flows are stable as $\alpha \to 0$ for bounded αR, and that their mechanism of instability is associated with the singularity of the critical layer where $w_*(y_*) = c_*$ when $\alpha R \to \infty$ for fixed α. This mechanism can never be described by our conditions derived for small L, i.e. for small α. Thus we shall apply our conditions to unbounded flows only, for which instability occurs at small α and finite αR, and not bother to verify the stability of bounded flows for this range of α and R.

We shall not consider the smoothly varying profile further in our analysis, so it is convenient to redefine dimensionless variables with H as length scale after taking the limit $L \to 0$. Thus, with $\alpha \equiv \alpha_* H$, $R \equiv VH/\nu$, $w(y) \equiv V^{-1} w_*(y_*/H)$, etc., henceforth, we get dimensionless conditions

$$[\phi] = 0, \tag{22}$$

$$[D\phi] = 0, \tag{23}$$

$$[D^2\phi + i\alpha R(w-c)\,\phi] = 0, \tag{24}$$

$$[D^3\phi - i\alpha R\{(w-c)\,D\phi - (Dw)\,\phi\}] = 0, \tag{25}$$

for bounded α, αR at a discontinuity of w and/or Dw.

Rayleigh's use (1945, ch. 21) of broken-line profiles for the stability equation

$$(w-c)\,(D^2-\alpha^2)\,\phi - (D^2 w)\,\phi = 0 \tag{26}$$

of an inviscid fluid can be justified similarly from the integrals

$$(w-c)\,D\phi - (Dw)\,\phi = \alpha^2 D^{-1}(w-c)\,\phi,$$

$$\phi/(w-c) = \alpha^2 D^{-1}(w-c)^{-2}\,D^{-1}(w-c)\,\phi.$$

It can be seen that

$$[(w-c)\,D\phi - (Dw)\,\phi] = 0, \tag{27}$$

$$[\phi/(w-c)] = 0 \tag{28}$$

at a discontinuity of w or Dw. In fact, these are the familiar conditions that pressure and normal velocity respectively are continuous at an interface. (Critical reviews of the validity of these conditions are given by Lin (1945, pp. 121, 221) and by Drazin & Howard (1961).)

3. Broken-line half-jet

If $w(y)$ is piecewise constant, the Orr–Sommerfeld equation has solutions

$$\phi = e^{\pm\alpha y}, \quad e^{\pm\beta y}, \tag{29}$$

where

$$\beta \equiv +\{\alpha^2 - i\alpha R(c-w)\}^{\frac{1}{2}} \tag{30}$$

has non-negative real part for definiteness. In this case the boundary conditions (22) to (25) become

$$[\phi] = 0, \tag{31}$$

$$[D\phi] = 0, \tag{32}$$

$$[(D^2+\beta^2)\,\phi] = 0, \tag{33}$$

$$[(D^2-\beta^2)\,D\phi] = 0. \tag{34}$$

The simplest flow is that in which w has one jump; this is the Helmholtz flow or broken-line half-jet with

$$w = y/|y| \quad (-\infty < y < \infty). \tag{35}$$

(In this case the basic flow has only one length scale L, but we may choose H arbitrarily for use in the dimensionless equations.) The most general solution of the Orr–Sommerfeld equation satisfying the boundary conditions at infinity is of the form

$$\phi = \begin{cases} A\,e^{-\alpha y} + B\,e^{-\beta_1 y} & (y > 0), \\ D\,e^{\alpha y} + E\,e^{\beta_2 y} & (y < 0), \end{cases} \tag{36}$$

for some constants A, B, D, E where

$$\beta_1 \equiv +\{\alpha^2 - i\alpha R(c-1)\}^{\frac{1}{2}}, \quad \beta_2 \equiv +\{\alpha^2 - i\alpha R(c+1)\}^{\frac{1}{2}}. \tag{37}$$

The boundary conditions (31) to (34) at $y = 0$ give four homogeneous linear equations in A, B, D, E. A non-zero solution exists if and only if their discriminant is zero; thus the eigenvalue relation is

$$\begin{aligned} 0 &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ -\alpha & -\beta_1 & \alpha & \beta_2 \\ \beta_1^2+\alpha^2 & 2\beta_1^2 & \beta_2^2+\alpha^2 & 2\beta_2^2 \\ \alpha(\beta_1^2-\alpha^2) & 0 & -\alpha(\beta_2^2-\alpha^2) & 0 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ -\alpha & -\beta_1 & \alpha & \beta_2 \\ 0 & \beta_1^2-\alpha^2 & \beta_2^2-\beta_1^2 & 2\beta_2^2-\beta_1^2-\alpha^2 \\ 0 & -\beta_1(\beta_1^2-\alpha^2) & \alpha(\beta_1^2-\beta_2^2) & \beta_2(\beta_1^2-\alpha^2) \end{vmatrix}. \end{aligned} \tag{38}$$

This last equation is $E_{0000} = 0$ in equation (4.2) of Tatsumi & Gotoh (1960). On evaluation of that determinant, they found

$$2\alpha(\beta_1-\alpha)(\beta_2-\alpha)(\beta_1+\beta_2)\{\beta_1^2+\beta_2^2-\beta_1\beta_2+\alpha(\beta_1+\beta_2)+\alpha^2\} = 0.$$

The only roots relevant to the flow are those of

$$\beta_1^2+\beta_2^2-\beta_1\beta_2+\alpha(\beta_1+\beta_2)+\alpha^2 = 0, \tag{39}$$

which is equation (4.8) of Tatsumi & Gotoh (1960). On division by α^2 it can be seen that this equation is reducible to the form (11), viz. $c = f(R/\alpha)$. On squaring (39) twice to eliminate radicals, we find a quadratic in R/α whose roots are

$$R/\alpha = -4i(c \pm 3^{\frac{1}{2}}i)/(3^{\frac{1}{2}}c \pm i)^2. \tag{40}$$

Therefore

$$c = \tfrac{1}{3} - 2i(\alpha/R) \pm 3^{\frac{1}{2}}i \pm 2\{-(\alpha/R)^2 \pm 2.3^{\frac{1}{2}}(\alpha/R)\}^{\frac{1}{2}}. \tag{41}$$

The first and third $\pm$ signs are ordered as those in the previous equation; the second is independent. It can be shown from equation (39) that the appropriate solution with $\mathrm{Re}\,\beta_1$, $\mathrm{Re}\,\beta_2 \geqslant 0$ is given by

$$c_r = 0, \tag{42}$$

and

$$c_i = \tfrac{1}{3}[3^{\frac{1}{2}} - 2(\alpha/R) - 2\{(\alpha/R)^2 + 2.3^{\frac{1}{2}}(\alpha/R)\}^{\frac{1}{2}}], \tag{43}$$

or

$$\alpha/R = (1-3^{\frac{1}{2}}c_i)^2/4(3^{\frac{1}{2}}-c_i). \tag{44}$$

This is the solution of Tatsumi & Gotoh.

It follows that

$$c \sim -4i\alpha/3R \quad \text{as} \quad R/\alpha \to 0, \tag{45}$$

and

$$c \to 3^{-\frac{1}{2}}i \quad \text{as} \quad R/\alpha \to \infty. \tag{46}$$

Also the curve of neutral stability ($c_i = 0$) is

$$R = 4.3^{\frac{1}{2}}\alpha, \tag{47}$$

as found by Esch (1957) and Tatsumi & Gotoh (1960).

4. Broken-line jet

Suppose
$$w = \begin{cases} 0 & (|y| > 1), \\ 1 & (|y| < 1). \end{cases} \tag{48}$$

The anti-symmetric disturbance with even $\phi(y)$ is thought (cf. Tatsumi & Kakutani 1958; Howard 1959; Clenshaw & Elliott 1960; Drazin & Howard 1961) to be the least stable for even profiles, so we shall take the most general even solution
$$\phi = \begin{cases} A\exp\{-\alpha(|y|-1)\}+B\exp\{-\beta_0(|y|-1)\} & (|y| > 1), \\ D\dfrac{\cosh\alpha y}{\cosh\alpha}+E\dfrac{\cosh\beta_1 y}{\cosh\beta_1} & (|y| < 1), \end{cases} \tag{49}$$
where
$$\beta_0 \equiv +\{\alpha^2-i\alpha Rc\}^{\frac{1}{2}}, \tag{50}$$
satisfying the Orr–Sommerfeld equation and the boundary conditions at infinity.

On elimination of A, B, D, E from the boundary conditions (31) to (34) at $y=\pm 1$, we find
$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ -\alpha & -\beta_0 & \alpha\tanh\alpha & \beta_1\tanh\beta_1 \\ \beta_0^2+\alpha^2 & 2\beta_0^2 & \beta_1^2+\alpha^2 & 2\beta_1^2 \\ \alpha(\beta_0^2-\alpha^2) & 0 & -\alpha(\beta_1^2-\alpha^2)\tanh\alpha & 0 \end{vmatrix} = 0. \tag{51}$$

If $H\to\infty$, we see that the jet has the same stability characteristics as two distantly separated half-jets, each with velocity difference unity, because equation (51) becomes (38) with β_0 for β_1 and β_1 for β_2 in the limits $\alpha, R\to\infty$.

In general, after discarding the trivial factors $\alpha(\beta_0-\alpha)(\beta_1^2-\alpha^2)$, we can reduce equation (51) to
$$\begin{aligned} \beta_0(\beta_0+\alpha)+(2\beta_1^2-\beta_0^2+2\alpha\beta_0+\alpha^2)\tanh\alpha+\alpha(\beta_0+\alpha)\tanh^2\alpha \\ +(\beta_0+\alpha)(\beta_1^2-\alpha^2)^{-1}(\beta_1\tanh\beta_1-\alpha\tanh\alpha) \\ \times\{2\beta_0^2-\beta_1^2-\alpha^2+(\beta_1^2-\alpha^2)\tanh\alpha\} = 0. \end{aligned} \tag{52}$$

When $\alpha R\to\infty$ for fixed α, the inviscid limit of equation (52) gives
$$3(1+\tanh\alpha)^2c^2+(1+\tanh\alpha)(1-7\tanh\alpha)c+4\tanh^2\alpha = 0.$$

This gives complex conjugate roots c when $7-4.3^{\frac{1}{2}} < \tanh\alpha < 7+4.3^{\frac{1}{2}}$; therefore there is instability only when $\alpha > \tanh^{-1}(7-4.3^{\frac{1}{2}})$. This differs from Rayleigh's result that $c=\{1\pm i(\coth\alpha)^{\frac{1}{2}}\}/(1+\coth\alpha)$ when $\alpha R\to\infty$ in the Orr–Sommerfeld equation before the broken-line profile is taken, i.e. when the broken-line profile is used with equation (26). It will be shown in § 5 that Rayleigh's result is the physically meaningful one, and that the result of this paragraph is useful only to elucidate the roots of equation (52) for finite αR and small α.

When $\alpha\to 0$ for fixed R/α in equation (52), it can be shown that
$$c = -i(\alpha/R)\{1+R^2+4R^2(\alpha+iR)+O(\alpha^4)\}.$$

This gives stability near the origin in the (α, R)-plane, in agreement with our heuristic conclusion (13).

When $R \to \infty$ for fixed $X \equiv \alpha R^2$, equation (52) gives

$$c = \alpha^2\{4(X^{-1}-\tfrac{1}{3})-4iX^{-1}R^{-1}(\tfrac{4}{45}X^2-3X+4)+O(R^{-2})\}. \tag{53}$$

Therefore $c_i(\alpha, R) = 0$ has asymptotes

$$\tfrac{4}{45}X^2-3X+4 = 0. \tag{54}$$

Thus, there are branches of the curve of neutral stability on which $\alpha R^2 \to 1{\cdot}34$ or $32{\cdot}4$ and $c \sim 1{\cdot}54\alpha^2$ or $-1{\cdot}21\alpha^2$, respectively, as $\alpha \to 0$. The relation (53) has been found by Tatsumi & Kakutani (1960)†, and is similar to that for the smoothly varying jet $w = \operatorname{sech}^2 y$.

If symmetric disturbances with odd $\phi(y)$ are considered instead of antisymmetric disturbances (49), it can be seen that the eigenvalue equation is the same as (52) except that hyperbolic cotangents replace hyperbolic tangents. Thus, in the limit $\alpha R \to \infty$ for fixed α, c has complex conjugate values and the flow is unstable if $\alpha > \coth^{-1}(7+4.3^{\frac{1}{2}})$. As $\alpha \to 0$ for fixed αR, equation (52), modified for symmetric disturbances, gives

$$\beta_0\beta_1+(2\beta_1^2-\beta_0^2)\tanh\beta_1 = O(\alpha).$$

This has limiting solution

$$c = 1+2n^2\pi^2/\alpha R,$$

$$\alpha R = 2n^2\pi^2/[1-\tfrac{3}{2}\cosh^2(n\pi)\{1-(1-\tfrac{8}{9}\operatorname{sech}^2(n\pi))^{\frac{1}{2}}\}] \quad (n = 1,2,3,\ldots). \tag{55}$$

This type of behaviour, viz. $c \to$ const., $\alpha R \to$ const. on the lower branch of the curve of neutral stability, has already been found by Clenshaw & Elliott (1960) for symmetric disturbances of the jet $w = \operatorname{sech}^2 y$. The precise result (55) for the broken-line jet is new. It may be noted that these results confirm the greater instability of antisymmetric disturbances.

5. Discussion

The use of discontinuous profiles is suitable for small wave-numbers, for which variations of velocity of a smoothly varying profile occur in a small fraction of a wavelength. This use is the antithesis of the W.K.B. approximation, in which it is assumed that the coefficients of an ordinary differential equation are slowly varying. We shall not pursue the W.K.B. approximation because it is valid only for large wave-numbers, for which it is already known that flows are stable.

The stability characteristics (43) of a vortex-sheet or broken-line half-jet give $c_i = 3^{-\frac{1}{2}}$, and therefore instability, when $R \to \infty$. It might be thought that the wavelength was the only length-scale in instability of a vortex-sheet, and that therefore this result should agree with Helmholtz's that $c_i = \pm 1$ for all

† In this unpublished letter the authors pointed out that they had recently found that terms of order $(\alpha R)^3$ should be included in equation (6.4) of their paper of 1958 for the calculation of the proper limit of c for a jet-like profile as $\alpha \to 0$. The term

$$i(\alpha R)^3\{-\tfrac{1}{2}(6\alpha-13\beta)-3\beta\log 2+\tfrac{1}{8}\pi^2(3\alpha-4\beta)-\tfrac{3}{2}(\alpha-\beta)(\log 2)^2\}$$

should be added to equation (7.3) of their paper of 1958 on the stability of the Bickley jet $w = \operatorname{sech}^2 y$. This gives $c/\alpha^2 \to 2(2X^{-1}-1)$ as $\alpha \to 0$, when $(\frac{1}{3}\pi^2-1)X^2-9X+8 = 0$, the result first found by Howard (1959). It gives $\alpha R^2 \to 0{\cdot}954$ and $c \sim 2{\cdot}19\alpha^2$ as $\alpha \to 0$ on the lower branch of the curve of neutral stability. They also extended their results of 1958 to other jet-like profiles. In particular they found the stability characteristics of the broken-line jet for small α; our equation (53) agrees with these.

wave disturbances of a vortex-sheet in an inviscid fluid. This superficial contradiction can be resolved by remembering that our results are not for a vortex-sheet but for a smoothly varying half-jet in a certain limit of small wave-number. We took $\alpha \to 0$ for fixed R/α and then let $R \to \infty$, and Helmholtz's result comes from taking $\alpha R \to \infty$ first. The former corresponds to the origin and the latter to infinity in the (α, R)-plane. For similar reasons we cannot expect our eigenvalues (53) for the broken-line jet to be related to Rayleigh's eigenvalues for the same jet in an inviscid fluid.

The use of discontinuous profiles in §§ 2–4 is a simple tool to find stability characteristics for bounded αR and small α, but cannot be used for more. This limitation means that the method is not suitable for bounded or semi-bounded flows, which are already known to be stable in that region of the (α, R)-plane.

Tatsumi & Kakutani (1960) have given our approximation (53) to the eigenvalue of an antisymmetric disturbance of the broken-line jet by their series in αR (Tatsumi & Kakutani 1958). We presume that the complete power series can be shown to give our eigenvalue relation (52). Our result (55) for the symmetric disturbance is in qualitative agreement with that of Clenshaw & Elliott (1960) for the Bickley jet $w = \operatorname{sech}^2 y$.

Equation (53) shows that there is instability between the two branches of the curve of neutral stability on which $\alpha R^2 \to 1{\cdot}54$ and $32{\cdot}4$ as $\alpha \to 0$, i.e. the flow is unstable for values of (α, R) between these two branches. This strange result is not due to the broken-line profile, because similar behaviour can be found for the smoothly varying jet $w = \operatorname{sech}^2 y$. Howard's (1959) results imply that

$$c_i = -2\alpha R^{-3}\{(\tfrac{1}{6}\pi^2 - 1)(\alpha R^2)^2 - 9(\alpha R^2) + 8 + O(R^{-2})\}$$

as $\alpha \to 0$. Again, this gives instability as $R \to \infty$ if $0{\cdot}954 < \alpha R^2 < 13{\cdot}0$ and stability if αR^2 lies outside that interval. The significance of these two roots does not appear to have been recognized by previous authors.

We have found three branches of the curve (or curves) of neutral stability for the broken-line jet as $R \to \infty$. On the lowest $\alpha R^2 \to 1{\cdot}54$, on the middle $\alpha R^2 \to 32{\cdot}4$, on the highest $\alpha \to \tanh^{-1}(7 - 4.3^{\frac{1}{2}})$. The flow is unstable above the highest branch and between the lower pair. There do not appear to be any other branches as $R \to \infty$, and it may be concluded that the middle joins up with the highest, and that $\alpha \to \infty$ on the other end of the lowest branch. This gives two curves of neutral stability, between which there is instability. However, the broken-line jet can tell us little of the analogous behaviour of the smoothly varying jet, because they do not have similar stability characteristics when $\alpha R \to \infty$ or α is not small. This discrepancy was anticipated in § 2, where it was pointed out that the proof of conditions (22)–(25) was invalid for infinite αR.

For the jet $w = \operatorname{sech}^2 y$, three branches of the curve of neutral stability are known at large Reynolds numbers. On the highest, $\alpha \to 2$ and $\phi \to \operatorname{sech}^2 y$ as $R \to \infty$. On the lower pair, $\alpha R^2 \to 0{\cdot}954$ or $13{\cdot}0$. There is instability directly below the highest branch and between the lower pair. I am indebted to Dr J. T. Stuart, who in a private letter conjectured the synthesis of the above properties shown schematically in figure 1. The lowest branch joins with the highest, as supposed by Tatsumi & Kakutani (1958), Howard (1959) and Clenshaw & Elliott (1960). However, the middle branch may return to infinity with asymptotic behaviour $\alpha R^p \to$ constant as $R \to \infty$ for $0 < p < 1$. This gives a 'minor' curve of neutral stability inside the 'major' curve found by previous authors, the region of instability being between the two curves. On the upper branch of the minor curve $\alpha R \to \infty$, so it might be expected that $\phi \to w$, $c \to 0$, the 'trivial' eigensolution of the stability equation (26) for an inviscid fluid when $\alpha = 0$. If this were so, the argument Howard (1959, p. 285) suggested as plausible would be false. He made the explicit assumption that an integration and the limit

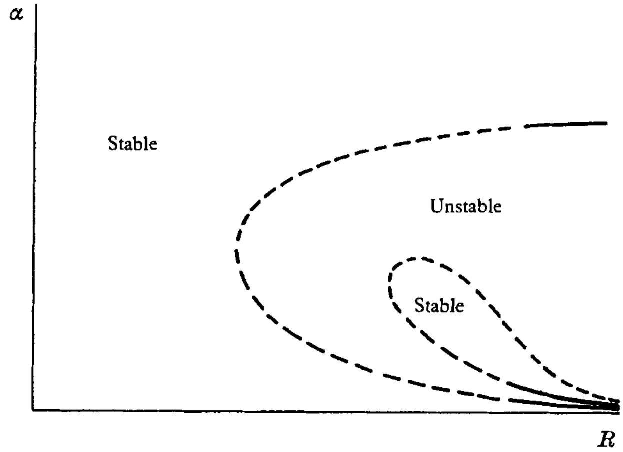

FIGURE 1. Sketch of the curve of neutral stability of the anti-symmetrical disturbance of a symmetrical smoothly varying jet. Conjectured parts of the curve are denoted by broken lines.

$R \to \infty$ could be inverted in order to show that the trivial solution was not a limit of the viscous eigensolution for unbounded flow; this inversion would be invalid if, for example, $\phi \sim w + R^{-1}\exp(-y^2/R^2)$ as $R \to \infty$, and now seems so for the actual eigensolution.

This conjecture of Stuart might be confirmed by the asymptotic theory of the Orr–Sommerfeld equation for large αR. However, the work of Tatsumi & Kakutani (1958), Howard (1959), and Clenshaw & Elliott (1960) shows clearly the occurrence and form of the most unstable disturbances of a jet, which will dominate the less unstable ones represented by points inside the 'minor' curve.

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